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# AN INVESTIGATION OF THE STABILITY OF NUMERICAL SOLUTIONS OF BIOT'S EQUATIONS OF CONSOLIDATION

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**Abstract**—In this paper a finite element formulation based on approximation in the Laplace transform space, is given for Biot's Consolidation theory. Various integration schemes are proposed and conditions under which these integration schemes are stable are investigated. The results are illustrated by means of a numerical example.

## INTRODUCTION

The theory of one dimensional consolidation was developed by Terzaghi[1]. Biot[2, 3] later extended the theory to three dimensional situations and subsequently[4, 5] modified his analysis to include the effects of anisotropy and visco-elasticity.

Biot's consolidation equations are quite complicated in that they combine the complexities of an elastic problem coupled with those of a diffusion process. For this reason it has been possible to devise analytic solutions to only the simplest problems[16, 17, 20] and for more complicated problems it has been necessary to devise numerical techniques.

Recently several investigators[6–13] have developed finite element formulations for the consolidation process, a convenient account is given by Sandhu[7]. The finite element equations are then solved by a marching technique. As is well known such marching techniques may well be unstable[14], however previous investigations seem incomplete in that they contain no investigation of the stability/instability of the proposed integration scheme. This considerably reduces the value of these formulations, for the stability of a proposed calculation can only be ascertained by numerical experiment for that particular problem and so in an extended calculation, the lack of criterion of convergence, makes it uncertain whether it is valid to increase the integration step size during the calculation.

## BASIC EQUATIONS

Biot's equations may be derived from the following considerations: (a) The stresses are in equilibrium. (b) The effective stresses are related to the strains through a generalized Hooke's Law. (c) The flow of water through the soil is governed by Darcy's Law. (d) The water is incompressible compared with the soil skeleton and thus the rate at which water flows out of an element equals the rate of volume change of that element.
This leads to the following equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \tag{1a}$$

$$\sigma'_{ij} = \sigma_{ij} + p\delta_{ij} = H_{ijkl}\epsilon_{kl} \tag{1b}$$

$$v_i = \frac{k_{ij}}{\gamma_\omega}\frac{\partial p}{\partial x_j} \tag{1c}$$

$$\frac{\partial v_i}{\partial x_i} = \frac{\partial \theta}{\partial t} \tag{1d}$$

$\sigma_{ij}$ are the components of the stress tensor (increase in total stress due to the applied tractions) tensile stresses are taken as positive.
$x_i$ is the position vector.

$t$ denotes the time.
$p$ is the excess pore pressure.
$u_i$ are the components of the displacement vector.
$\sigma'_{ij}$ are the components of the effective stress tensor.
$\epsilon_{ij}$ are the components of the strain tensor.
$H_{ijkl}$ are the elastic coefficients in the generalized Hooke's Law $H_{ijkl} = H_{jikl} = H_{klij}$.
$k_{ij}$ are the coefficients of permeability in the generalized Darcy's law $k_{ij} = k_{ji}$.
$v_i$ are the components of the superficial velocity vector of the pore water.
$\theta$ is the volume strain.

For definiteness the problem shown schematically in Fig. 1 will be considered. The soil

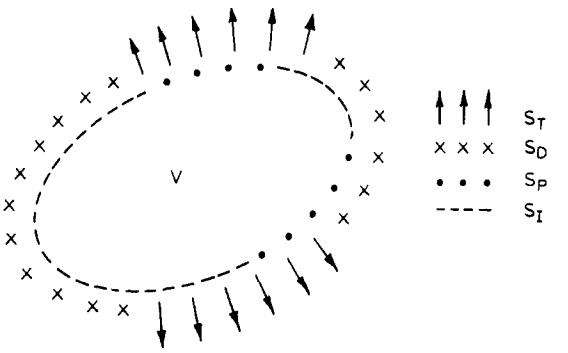


Fig. 1.

occupies a volume $V$; the portion of the surface $S_T$ is acted upon by applied tractions which may be assumed to have been applied instantaneously at $t = 0+$, the remainder of the surface $S_D$ is subjected to fixed zero displacements. The portion of the surface $S_p$ is assumed free to drain while the remainder $S_I$ is impermeable.†

Thus for this problem equation (1) must be solved subject to the boundary conditions:

$$\sigma_{ij}n_i = T_i \quad \text{on} \quad S_T \tag{2a}$$

$$u_i = 0 \quad \text{on} \quad S_D \tag{2b}$$

$$p = 0 \quad \text{on} \quad S_p \tag{2c}$$

$$n_i v_i = 0 \quad \text{on} \quad S_I \tag{2d}$$

where $n_i$ denotes the outward normal to the surface of the body.

The pore water is assumed incompressible, it therefore follows that, initially when the load is applied, there can be no instantaneous volume change and thus:

$$\theta = 0 \quad \text{when} \quad t = 0+ \tag{3}$$

It can be shown, by a slight extension of the method developed in [15], that the solution $(u_i, p)$ of equations (1–3) is the one which satisfies the conditions (2b, c) and minimizes the functional:

$$\Phi(u_i, p) = \int_V \left\{\frac{1}{2}H_{ijkl}\bar{\epsilon}_{ij}\bar{\epsilon}_{kl} - \bar{p}\bar{\theta} - 1/(2\gamma_\omega s)k_{ij}\frac{\partial\bar{p}}{\partial x_i}\frac{\partial\bar{p}}{\partial x_j}\right\} dV - \int_{S_T} \bar{T}_i\bar{u}_i \, dS \tag{4}$$

where a bar denotes the Laplace transform

$$\bar{u}_i = \int_0^\infty u_i \, e^{-st} \, dt \quad \text{etc.}$$

†More complicated boundary conditions, both elastic and hydraulic, are easily incorporated into the theory. The extensions are straightforward and will not be given here.

The minimization problem described by equation (4) may be solved approximately by the finite element technique. In doing this it is convenient to introduce the more standard notation: $\boldsymbol{\sigma}^T = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12})$ is the vector of stress components; $\boldsymbol{\epsilon}^T = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{23}, 2\epsilon_{31}, 2\epsilon_{12})$ is the vector of strain components; $\mathbf{D}$ is the matrix of elastic constants, so that equation (1b) becomes $\boldsymbol{\sigma}' = \mathbf{D}\boldsymbol{\epsilon}$; $\mathbf{k}$ is the matrix of permeabilities; so that equation (1c) becomes $\mathbf{v} = \mathbf{k}\nabla p$.

In this notation equation (4) becomes

$$\Phi(\bar{\mathbf{u}}, \bar{p}) = \frac{1}{2}\int \left| \bar{\boldsymbol{\epsilon}}^T \mathbf{D}\bar{\boldsymbol{\epsilon}} - 2\bar{p}\bar{\theta} - \frac{1}{s}\nabla\bar{p}^T \mathbf{k}\nabla\bar{p} \right| \mathrm{d}V - \int_{S_T} \bar{\mathbf{u}}^T \bar{\mathbf{T}}\, \mathrm{d}s \tag{5}$$

The approximate solution of equation (5) by finite element techniques is straight forward.

(1) Suppose that $\mathbf{u}, p$ can be adequately represented by their values at the nodes 1, 2, 3 . . .

$$\boldsymbol{\delta}^T = (\mathbf{u}_1^T; \mathbf{u}_2^T \ldots)$$
$$\mathbf{q}^T = (p_1, p_2 \ldots)$$

(2) Suppose that the continuous values of $\mathbf{u}$, $p$ may be approximated in terms of the nodal values.†

$$\mathbf{u} = \mathbf{C}(\mathbf{x})\boldsymbol{\delta}$$
$$p = \mathbf{a}^T(\mathbf{x})\mathbf{q}$$

(3) Express the approximations to strains, volume strain and pore pressure gradients in terms $\boldsymbol{\delta}$, $\mathbf{q}$

$$\boldsymbol{\epsilon} = \mathbf{B}(\mathbf{x})\boldsymbol{\delta}$$
$$\theta = \mathbf{d}^T(\mathbf{x})\boldsymbol{\delta}$$
$$\nabla p = \mathbf{E}(\mathbf{x})\boldsymbol{\delta}$$

where

$$\mathbf{B} = \begin{bmatrix} \partial/\partial x_1; & 0; & 0 \\ 0; & \partial/\partial x_2; & 0 \\ 0; & 0; & \partial/\partial x_3 \\ 0; & \partial/\partial x_3; & \partial/\partial x_2 \\ \partial/\partial x_3; & 0; & \partial/\partial x_1 \\ \partial/\partial x_2; & \partial/\partial x_1; & 0 \end{bmatrix} \mathbf{C}$$

$$\mathbf{E} = \begin{bmatrix} \partial \mathbf{a}^T/\partial x_1 \\ \partial \mathbf{a}^T/\partial x_2 \\ \partial \mathbf{a}^T/\partial x_3 \end{bmatrix}$$

$$\mathbf{d}^T = (1, 1, 1, 0, 0, 0)\mathbf{B}$$

(4) $\Phi$ may now be approximated in the form

$$\Phi_{\text{approx.}} = \tfrac{1}{2}(\bar{\boldsymbol{\delta}}^T \mathbf{K}\bar{\boldsymbol{\delta}} - 2\bar{\boldsymbol{\delta}}^T \mathbf{L}^T\bar{\mathbf{q}} - \bar{\mathbf{q}}^T\mathbf{M}\bar{\mathbf{q}}) - \bar{\boldsymbol{\delta}}^T\bar{\mathbf{b}}$$

$$\text{where } \mathbf{K} = \int_V \mathbf{B}^T\mathbf{D}\mathbf{B}\, \mathrm{d}V = \text{Elastic Stiffness Matrix} \tag{6a}$$

$$\mathbf{L} = \int_V \mathbf{a}\mathbf{d}^T\, \mathrm{d}V \tag{6b}$$

†C(x), a(x) are both known, their precise form will depend upon the particular finite element adopted.

$$\mathbf{M} = \int_V \frac{1}{\gamma_\omega} \mathbf{E}^T \mathbf{k} \mathbf{E} \, dV \tag{6c}$$

$$\bar{\mathbf{b}} = \int_{S_T} \mathbf{C}^T \bar{\mathbf{T}} \, dS \tag{6d}$$

Minimization of $\Phi_{approx.}$ leads to the set of equations

$$\mathbf{K}\bar{\boldsymbol{\delta}} - \mathbf{L}^T \bar{\mathbf{q}} = \bar{\mathbf{b}} \tag{7a}$$

$$-\mathbf{L}\bar{\boldsymbol{\delta}} - \frac{1}{s}\mathbf{M}\bar{\mathbf{q}} = \mathbf{0} \tag{7b}$$

Equation (7a, b) may now be inverted, and thus

$$\mathbf{K}\boldsymbol{\delta} - \mathbf{L}^T \mathbf{q} = \mathbf{b} \tag{8a}$$

$$\mathbf{L}\boldsymbol{\delta} - \mathbf{M}\int_0^t \mathbf{q} \, dt = \mathbf{0} \tag{8b}$$

It is often convenient to write equation (8b) in the alternative form

$$\frac{\mathbf{L}\, d\boldsymbol{\delta}}{dt} - \mathbf{M}\mathbf{q} = \mathbf{0} \tag{8c}$$

in this case it is necessary to append the condition

$$\mathbf{L}\boldsymbol{\delta} = \mathbf{0} \quad \text{when} \quad t = 0+ \tag{8d}$$

which is of course the finite element equivalent of equation (3).

It is perhaps worthwile at this stage to note that the initial/final solutions can be obtained by performing an elastic analysis and using the undrained/drained elastic constants and initial/final loads, respectively. A proof of this for this isotropic case is given in [19]: the extension to the anisotropic case provides no difficulty. These initial and final solutions provide useful independent checks upon the accuracy of approximate solutions.

## THE EXACT SOLUTION OF THE FINITE ELEMENT EQUATIONS

It will be useful, in obtaining the stability criterion for approximate solutions, to find an exact solution of the finite element equations (7, 8). This solution is the finite element analogue of previous analytic results [15].

Equations (7) may be reduced to a more conventional form, by using equation (7a) to express $\bar{\boldsymbol{\delta}}$ in terms of $\bar{\mathbf{q}}$ as follows

$$\bar{\boldsymbol{\delta}} = \bar{\boldsymbol{\delta}}_e + \mathbf{K}^{-1}\mathbf{L}^T \bar{\mathbf{q}} \tag{9a}$$

where $\bar{\boldsymbol{\delta}}_e = \mathbf{K}^{-1}\bar{\mathbf{b}}$ is the Laplace transform of the deflection vector due to the applied loads if no excess pore pressures were to develop, i.e. the soil were infinitely permeable or completely dry. Notice if the applied loads are independent of time $\bar{\boldsymbol{\delta}}_e$ will also be the Laplace transform of the final deflections.

When equation (9a) is substituted into equation (7b) it is found that

$$(\mathbf{M} + s\mathbf{P})\bar{\mathbf{q}} = s\bar{\mathbf{f}} \tag{9b}$$

where

$$\mathbf{P} = \mathbf{L}\mathbf{K}^{-1}\mathbf{L}^T \tag{9c}$$

and

$$\bar{\mathbf{f}} = -\mathbf{L}\bar{\boldsymbol{\delta}}_e \tag{9d}$$

Equation (9b) is of the classical eigenvalue form and thus, as is well known, the solution may be written:

$$\bar{\mathbf{q}} = \sum_{n=1}^{N} \frac{s}{s - s_n} \bar{a}_n \mathbf{Q}_n \tag{10a}$$

$$\bar{\boldsymbol{\delta}} = \bar{\boldsymbol{\delta}}_E + \sum_{n=1}^{N} \frac{s}{s - s_n} \bar{a}_n \boldsymbol{\Delta}_n \tag{10b}$$

where

$$\bar{a}_n = \mathbf{Q}_n^T \bar{\mathbf{f}} / (\mathbf{Q}_n^T \mathbf{P} \mathbf{Q}_n) \tag{10c}$$
$$\boldsymbol{\Delta}_n = \mathbf{K}^{-1} \mathbf{L}^T \mathbf{Q}_n$$

and where $s = s_n$, $n = 1, \ldots N$ are the eigenvalues of equation 9b)

$$\det (\mathbf{M} + s_n \mathbf{P}) = 0 \quad n = 1, \ldots N \tag{10d}$$

and $\mathbf{Q}_n$ are the corresponding eigenvectors.

$$(\mathbf{M} + s_n \mathbf{P}) \mathbf{Q}_n = \mathbf{0} \tag{10e}$$

It is not desirable in numerical calculations to form the inverse of $\mathbf{K}$, this may be overcome by writing equation (7) in the form

$$\mathbf{Z}(s) \bar{\mathbf{r}} = \bar{\mathbf{c}}$$

where

$$\mathbf{Z}(s) = \begin{bmatrix} \mathbf{K} & -\mathbf{L}^T \\ -\mathbf{L} & -\frac{1}{s}\mathbf{M} \end{bmatrix}; \qquad \bar{\mathbf{r}} = \begin{bmatrix} \bar{\boldsymbol{\delta}} \\ \bar{\mathbf{q}} \end{bmatrix}$$

and

$$\bar{\mathbf{c}} = \begin{bmatrix} \bar{\mathbf{b}} \\ \bar{\mathbf{0}} \end{bmatrix}.$$

The eigenvalues $s_n$ are then the zeros of

$$\det (\mathbf{Z}(s)) = 0$$

and equations (10a, b) may be written:

$$\bar{\mathbf{r}} = \bar{\mathbf{r}}_e + \sum_{n=1}^{N} \frac{s}{s - s_n} \bar{a}_n \mathbf{R}_n \tag{11a}$$

where

$$\mathbf{r}_e^T = (\boldsymbol{\delta}_e^T, \mathbf{0}) \tag{11b}$$

and

$$\mathbf{R}_n^T = (\mathbf{Q}_n^T, \boldsymbol{\Delta}_n^T) \tag{11c}$$

are the solutions of

$$\mathbf{Z}(s_n) \mathbf{R}_n = \mathbf{0}.$$

In terms of these eigenvalues the coefficients $\bar{a}_n$ are best calculated from the expression:

$$\bar{a}_n = (\mathbf{Q}_n^T \mathbf{L}^T \bar{\boldsymbol{\delta}}_E)/(\mathbf{Q}_n^T \mathbf{L}^T \boldsymbol{\Delta}_n).$$

Equation (11a) may now be inverted to give

$$\mathbf{r} = \mathbf{r}_e + \sum_{n=1}^{N} \mathbf{R}_n \left\{ a_n(0) \cdot e^{s_n t} + \int_0^t \frac{da_n(t')}{dt'} e^{s_n(t-t')} dt' \right\}$$

and

$$a_n(t) = (\mathbf{Q}_n^T \mathbf{L}^T \boldsymbol{\delta}_e(t))/(\mathbf{Q}_n^T \mathbf{L} \boldsymbol{\Delta}_n).$$

As would be expected from physical reasoning the eigenvalues $s_n$ are all negative. This follows since $s_n$ may be written in the form:

$$s_n = -(\mathbf{Q}_n^T \mathbf{M} \mathbf{Q}_n)/(\mathbf{Q}_n^T \mathbf{P} \mathbf{Q}_n)$$

and both **M**, **P** are positive definite, equations (6a, c; 9c).

It is often useful to know the values of field quantities at small and large times, these may be obtained by physical reasoning as indicated in the previous section or by means of the Tauberian theorems for Laplace transforms [18]. Briefly these theorems state that if $x(t)$ has the Laplace transform $\bar{x}(s)$ then:

$$\lim_{t \to 0+} x(t) = \lim_{s \to \infty} s\bar{x}(s)$$

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s\bar{x}(s)$$

provided these limits exist.

Applying these results to equation (7a, b) it may be seen that the initial solution, denoted by a subscript $I$, satisfies the equations:

$$\begin{aligned} \mathbf{K}\boldsymbol{\delta}_I - \mathbf{L}^T \mathbf{q}_I &= \mathbf{b}_I \\ -\mathbf{L}\boldsymbol{\delta}_I &= \mathbf{0} \end{aligned} \tag{12}$$

Similarly if the final solution is indicated by a subcript $F$, then

$$\begin{aligned} \mathbf{K}\boldsymbol{\delta}_F &= \mathbf{b}_F \\ \mathbf{q}_F &= \mathbf{0}. \end{aligned} \tag{13}$$

It is easily shown that Eq. (12, 13) are precisely those which would be used in an undrained, drained elastic analysis, as indicated in the previous section.

## APPROXIMATE SOLUTION OF THE FINITE ELEMENT EQUATIONS

The method developed in the previous section is quite attractive for solutions at large or moderate times, however for earlier times it is more convenient to use conventional 'marching' techniques. These techniques have the advantage of being applicable in situations where the eigenvalue technique is inapplicable, i.e. in problems involving moving boundaries, that is in problems in which there is addition or removal of elements.

An approximate solution of equation (8), can be found by integrating equation (8c) from $t$ to $t + \Delta t$, or what is equivalent by subtracting equation (8b) evaluated at $t$, from its value at $t + \Delta t$, and making the following approximation:

$$\int_t^{t+\Delta t} \mathbf{q}(t')\, dt' \doteq \Delta t(\alpha \mathbf{q}(t) + (1-\alpha)\mathbf{q}(t+\Delta t)). \tag{14}$$

In equation (14) a particular value of $\alpha$ will correspond to a particular integration rule, e.g. $\alpha = \frac{1}{2}$ corresponds to trapezoidal integration.

Equations (8) now become

$$\begin{aligned} \mathbf{K}\boldsymbol{\delta}(t+\Delta t) - \mathbf{L}^T\mathbf{q}(t+\Delta t) &= \mathbf{b}(t+\Delta t) \\ -\mathbf{L}\boldsymbol{\delta}(t+\Delta t) - (1-\alpha)\Delta t\mathbf{M}\mathbf{q}(t+\Delta t) &= \mathbf{c}[\boldsymbol{\delta}(t), \mathbf{q}(t)] \end{aligned} \tag{15}$$

where

$$\mathbf{c}[\boldsymbol{\delta}(t), \mathbf{q}(t)] = -\mathbf{L}\boldsymbol{\delta}(t) + \alpha\Delta t\mathbf{M}\mathbf{q}(t).$$

Clearly equation (15) can be used to obtain the value of $\boldsymbol{\delta}, \mathbf{q}$ at $t + \Delta t$ from their own values at $t$; and thus the values of $\boldsymbol{\delta}$, $\mathbf{q}$ may be found at all times by marching forward.

More formally, suppose it is desired to evaluate the solution at times $t = 0, \Delta t, 2\Delta t \ldots$. Suppose that the value of a given quantity at time $t = i\Delta t$ is designated by the subscript $i$, then;

$$\mathbf{K}_c((1-\alpha)\Delta t)\mathbf{r}_{i+1} = \mathbf{f}_{i+1} \quad i = 0, \ldots, m. \tag{16}$$

where

$$\mathbf{K}_c((1-\alpha)\Delta t) = \begin{bmatrix} \mathbf{K}, & -\mathbf{L}^T \\ -\mathbf{L}, & -(1-\alpha)\Delta t\mathbf{M} \end{bmatrix} = \mathbf{Z}\left(\frac{1}{(1-\alpha)\Delta t}\right)$$

is the "consolidation stiffness matrix".

$$\mathbf{r}_{i+1} = \begin{bmatrix} \boldsymbol{\delta}_{i+1} \\ \mathbf{q}_{i+1} \end{bmatrix}$$

and $\mathbf{f}_{i+1}$ denotes the "loadset"

$$\mathbf{f}_{i+1} = \begin{bmatrix} \mathbf{b}_{i+1} \\ -\mathbf{L}\boldsymbol{\delta}_i + \alpha\Delta t\mathbf{M}\mathbf{q}_i \end{bmatrix}.$$

The initial solution may be found by solving equation (12) or what is equivalent by solving

$$\mathbf{K}_c(0)\mathbf{r}_1 = \mathbf{f}_1$$

where

$$\mathbf{f}_I^T = (\mathbf{b}_I, \mathbf{0}). \tag{17}$$

Notice that $\mathbf{K}_c$ is constant and depends only on the assumed constant value of $(1-\alpha)\Delta t$. This observation may be used to conserve computing time, for successive vectors $\mathbf{f}_{i+1}$ may be treated as different load sets on some prescribed "structure". Thus, if equation (16) is solved by Crout-Cholesky factorization, then the matrix $\mathbf{K}_c$ need only be factored once.

In practice it is usually desirable to change the step size during the calculation in such cases $\mathbf{K}_c((1-\alpha)\Delta t)$ will alter (except when $\alpha = 1$) and so a new factorization must be performed.

Superficially it would seem that there is some merit in choosing $\alpha = 1$, as this matrix will presumably have been factored in obtaining the initial solution and $\mathbf{K}_c$ will remain fixed no matter what step size is adopted, and so no new factorization would be required. It be shown in the following section that such a process is unstable and so the choice of $\alpha = 1$ is not as attractive as it would first seem.

## STABILITY OF THE INTEGRATION SCHEME

As indicated in the introduction a numerical integration scheme involving forward marching will only be of value provided that it is unconditionally stable or the conditions for stability are known in advance.

The stability of the recurrence relation (16) will depend upon its homogeneous form:

$$\mathbf{K}\boldsymbol{\delta}_{t+1} - \mathbf{L}^T\mathbf{q}_{t+1} = \mathbf{0} \tag{18a}$$

$$-\mathbf{L}\{\boldsymbol{\delta}_{t+1} - \boldsymbol{\delta}_t\} - \mathbf{M}\{(1-\alpha)\mathbf{q}_{t+1} + \alpha\mathbf{q}_t\}\Delta t = \mathbf{0}. \tag{18b}$$

Equation (18a) may be used to eliminate both $\boldsymbol{\delta}_{t+1}$, $\boldsymbol{\delta}_t$ from equation (18b) and so on

$$\mathbf{P}\{\mathbf{q}_{t+1} - \mathbf{q}_t\} - \mathbf{M}\{(1-\alpha)\mathbf{q}_{t+1} - \alpha\mathbf{q}_t\}\Delta t = \mathbf{0}. \tag{18c}$$

Now if $\mathbf{q}_t$ is expanded in terms of the eigenvectors $\mathbf{Q}_n$, so that

$$\mathbf{q}_t = \sum_{j=1}^{N} a_{t,j}\mathbf{Q}_j$$

then it is found that the coefficients $a_{ij}$ satisfy the simple recurrence relation

$$a_{i+1,j} = \left(\frac{1+\alpha s_j\Delta t}{1-(1-\alpha)s_j\Delta t)}\right)a_{i,j}.$$

Now recalling that the eigenvalues $s_j$ are all negative and writing $s_j = -\beta_j^2$, this equation becomes

$$\frac{a_{i+1,j}}{a_{i,j}} = \frac{1-\alpha\beta_j^2\Delta t}{1+(1-\alpha)\beta_j^2\Delta t}.$$

It therefore follows that the proposed integration scheme is stable provided:

$$\left|\frac{1-\alpha\beta_j^2\Delta t}{1+(1-\alpha)\beta_j^2\Delta t}\right| \leq 1 \qquad j = 1, \ldots N.$$

On analyzing this inequality it may be seen that if $\alpha > \frac{1}{2}$ then the process is stable provided:

$$\Delta t \leq \min_j \{1/\beta_j^2(\alpha - \tfrac{1}{2})\}$$

or

$$\Delta t \leq 1/\beta_N^2(\alpha - \tfrac{1}{2}).$$

While if $\alpha \leq \frac{1}{2}$ the process is unconditionally stable.

In order to apply the stability criterion when $\alpha > \frac{1}{2}$ it is necessary to calculate the eigenvalue of greatest magnitude $s_N = -\beta_N^2$. Although this may be done by standard techniques, it is usually more convenient to circumvent this calculation and choose $\alpha < \frac{1}{2}$.

Some investigators[12, 13] have found it convenient to increase the step size $\Delta t$ throughout the solution process. Thus if

$$\Delta t_i = t_{i+1} - t_i,$$

equation (18c) becomes

$$\mathbf{P}(\mathbf{q}_{t+1} - \mathbf{q}_t) - \mathbf{M}\{(1-\alpha)\mathbf{q}_{t+1} + \alpha\mathbf{q}_t\}\Delta t = \mathbf{0}$$

and in this case

$$a_{i+1,j} = \frac{1+\alpha s_j\Delta t_i}{1-(1-\alpha)s_j\Delta t_i}a_{i,j}$$

so that

$$a_{i+1,j} = a_{0,j}\prod_{k=1}^{i+1}\left\{\frac{1+\alpha s_j\Delta t_k}{1-(1-\alpha)s_j\Delta t_k}\right\}.$$

It now becomes evident that, if $\{\Delta t_i\}$ is a strictly increasing sequence, the numerical solution will always be unstable whenever $\alpha > \frac{1}{2}$ and will be unconditionally stable if $\alpha \leq \frac{1}{2}$.

## NUMERICAL EXAMPLE

The following example, shown schematically in Fig. 2a, will be used to illustrate the conclusions of the previous section.

In Fig. 2a, an isotropic consolidating material is compressed, under conditions of plane strain, by a constant load $P$, between smooth rigid impermeable plates AA′, BB′. The boundaries AB, A′B′ are stress free and free to drain.

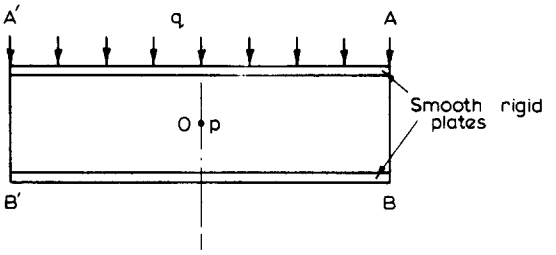


FIG. (2a)

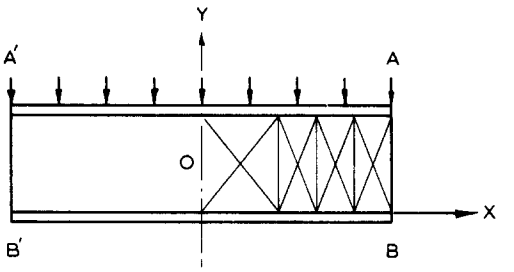


FIG. (2b)

In obtaining a numerical solution to this problem the arrangement of elements shown in Fig. 2b (triangles with linear variation of displacement and excess pore pressure) was used.

For definiteness attention was restricted to the following values of physical and geometric parameters:

$a$ = half the width of the impermeable plates = 2 units

$E$ = Young's modulus = 1 unit.

$\nu$ = Poisson's ratio = 0·25.

$k/\gamma_w$ = Coefficient of permeability = 1 unit/unit weight of water

For these values it was found that the greatest eigenvalue was:

$$\beta_N^2 = 1{\cdot}629 \times 10^4$$

and thus that:

(i) $\alpha \leq \frac{1}{2}$ process is always stable

(ii) $\alpha > \frac{1}{2}$ process is stable provided

$$\Delta t \leq 0{\cdot}614 \times 10^{-4}/(\alpha - 0{\cdot}5). \tag{19}$$

The effect of variation of $\alpha$ with a constant time step is shown in Fig. 3. In this figure the central pore pressure is plotted against dimensionless time $T_V$ ($T_V = 1/3\ (E/1+\nu)(1-\nu)/(1-2\nu)kt/a^2$) for a series of values of $\alpha$, when $\Delta t = 2{\cdot}04 \times 10^{-4}$. Equation (19) indicates that the calculation is stable for $\alpha \leq 0{\cdot}8$. This conclusion is borne out by Fig. 3, it can be seen that

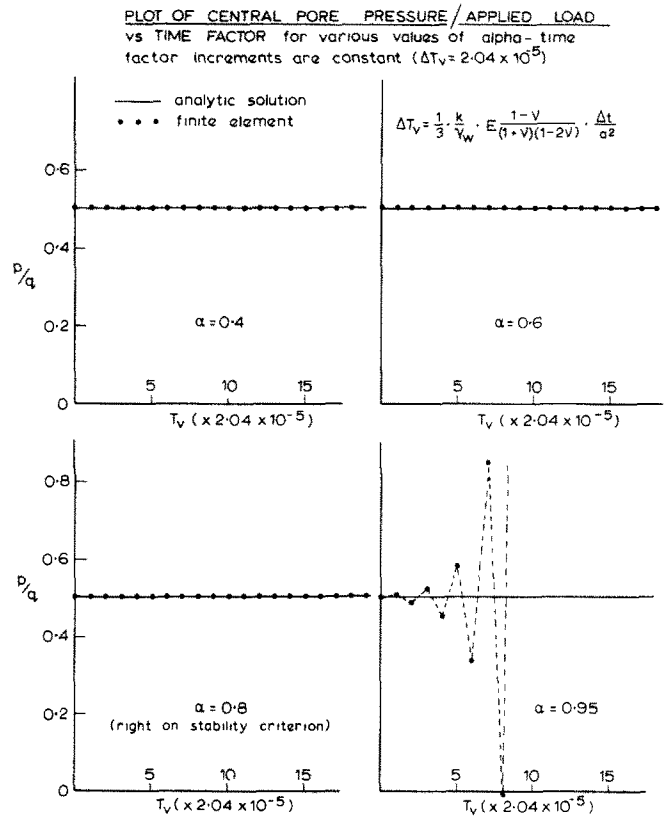
PLOT OF CENTRAL PORE PRESSURE / APPLIED LOAD
vs TIME FACTOR for various values of alpha - time factor increments are constant ($\Delta T_v = 2{\cdot}04 \times 10^{-5}$)
analytic solution
finite element
$\Delta T_v = \frac{1}{3} \cdot \frac{k}{\gamma_w} \cdot E \frac{1-\nu}{(1+\nu)(1-2\nu)} \cdot \frac{\Delta t}{a^2}$
p/q
$\alpha = 0{\cdot}4$
$\alpha = 0{\cdot}6$
$\alpha = 0{\cdot}8$
(right on stability criterion)
$\alpha = 0{\cdot}95$
$T_v$ ( x 2·04 x 10$^{-5}$)

Fig. 3.

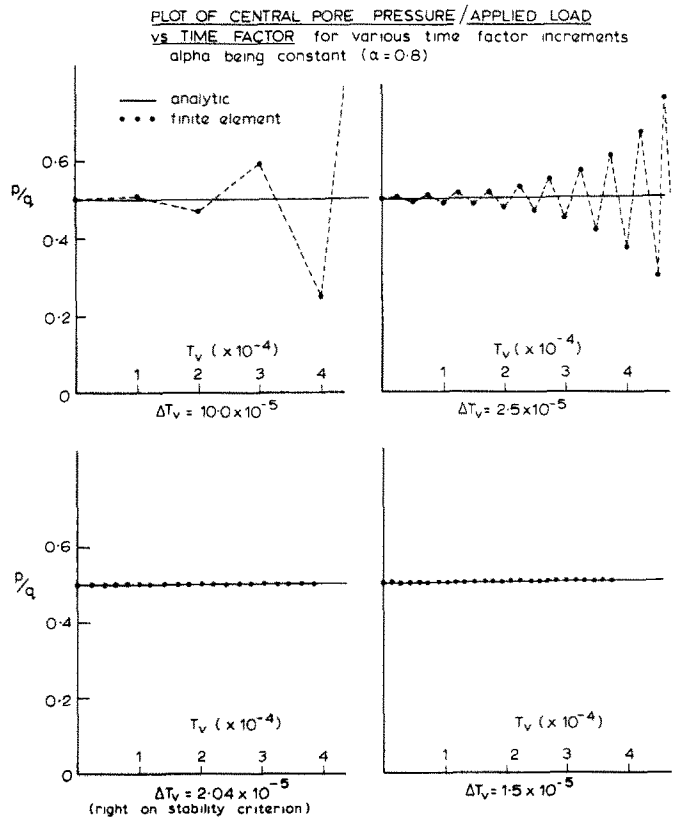
PLOT OF CENTRAL PORE PRESSURE / APPLIED LOAD
vs TIME FACTOR for various time factor increments alpha being constant ($\alpha = 0{\cdot}8$)
analytic
finite element
p/q
$T_v$ ( x 10$^{-4}$)
$\Delta T_v = 10{\cdot}0 \times 10^{-5}$
$\Delta T_v = 2{\cdot}5 \times 10^{-5}$
$\Delta T_v = 2{\cdot}04 \times 10^{-5}$
(right on stability criterion)
$\Delta T_v = 1{\cdot}5 \times 10^{-5}$

Fig. 4.

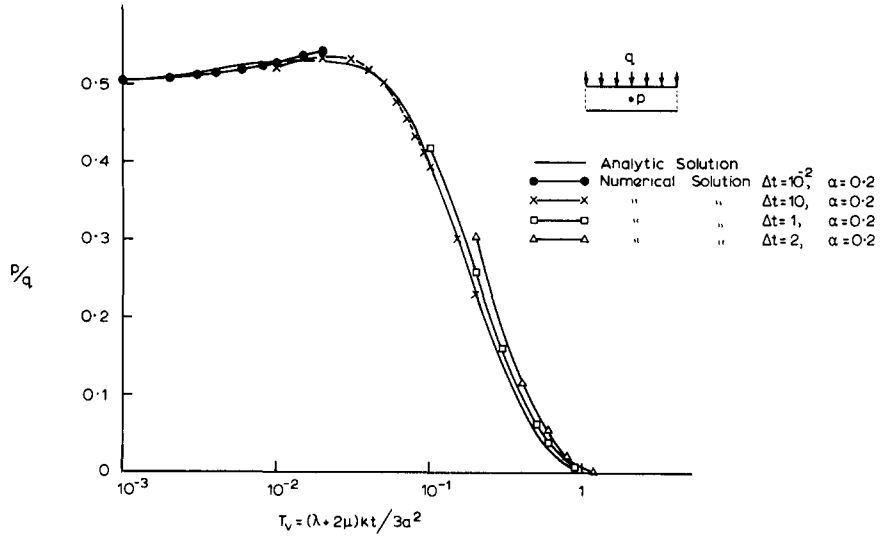


Fig. 5.

when $\alpha > 0{\cdot}8$ the numerical solution is unstable while for $\alpha \leq 0{\cdot}8$ the solution is in good agreement with the analytic solution[20].

The effect of variation with time step is illustrated in Fig. 4 for the particular case of $\alpha = 0{\cdot}8$. Inequality (15) predicts that the calculation will be stable provided

$$\Delta t \leq 2{\cdot}04 \times 10^{-4}$$

and again this is borne out by the numerical results.

Finally the effect of varying the step size for a case which is always stable ($\alpha = 0{\cdot}2$) is shown in Fig. 5. The numerical results are in quite good agreement with the analytic solution[20] even for quite large time steps.

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